Chapter 4:

Higher Order Linear Equations:

Section 4.1: General theory of nth order linear equations:

Recall that An n-th order linear ~~differ~~ ODE is an equation of the form:

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1} y}{dt^n} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)\,y = G(t). \quad (*)$$

- We can write this equation in terms of linear operators: (We divide the equation by $P(t)$, we obtain

$$L[y] = y^{(n)}(t) + p_1(t)\cdot y^{(n-1)}(t) + \cdots + p_{n-1}(t)\,y'(t) + p_n(t)\cdot y(t) = g(t).$$

It's natural to expect that to obtain a unique solution, it is necessary to specify n initial conditions

$$y(t_0)=y_0,\ y'(t_0)=y_0',\ \cdots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}.$$

Theorem: If the functions $p_1, p_2, \cdots, p_n$ and g are continuous on the open interval I, then there exists exactly one solution $y=\phi(t)$ of the equation w/ initial conditions, where t_0 is any point in I. This solution exists throughout the interval I.

The homogeneous equation: $g(t) \equiv 0$.

$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \cdots + p_{n-1}(t)y' + p_n(t)y = 0 \qquad (*)$$

If the functions $y_1, y_2, \cdots, y_n$ are solutions of $(*)$, then their linear combination

$$y = c_1 y_1(t) + \cdots + c_n y_n(t),$$ are also solutions of the homogeneous equations.

The natural question is: If all the solutions of the homogeneous equations are of the above form;

This will be true, if for ~~all~~ any choice of the pair t_0 in I, and for any choice of $y_0, y_0', \cdots, y_0^{(n-1)}$, we must be able to determine $c_1, \cdots, c_n$ so that:

$$c_1 y_1(t_0) + \cdots + c_n y_n(t_0) = y_0$$
$$c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) = y_0'$$
$$\vdots$$
$$c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = y_0^{(n-1)}$$

are satisfied.

The ~~matrix~~ determine of the matrix

$$W(y_1, \cdots, y_n)(t_0) \stackrel{def}{=} \begin{pmatrix} y_1(t_0) & \cdots & y_n(t_0) \\ \vdots & & \\ y_1^{(n+1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix}$$

is called the Wronskian.

~~$W(y_1, \cdots, y_n) =$~~

Theorem: If the function $p_1, p_2 \cdots, p_n$ are continuous on the interval I, and if $y_1, y_2, \cdots, y_n$ are solutions of the equation

$$y^{(n)}(t) + p_1(t) y^{(n-1)}(t) + \cdots + p_{n-1}(t) y'(t) + p_n(t) \cdot y(t) = 0 \quad (*),$$

and if $W(y_1, y_2, \cdots, y_n)(t) \neq 0$ for at lease one point in I, then every solution of $(*)$ can be expressed as a linear combination of the solutions ~~of~~ $y_1, \cdots, y_n$

Definition: $y_1, \cdots, y_n$ is called a fundamental set of solutions of the equation $(*)$.

Linear dependence and Wronskian:

Definition: The functions $f_1, \cdots, f_n$ are said to be linearly dependence on an interval I if there exists a set of constants $k_1, k_2, \cdots, k_n$, not all zero, such that

$$k_1 f_1(t) + \cdots + k_n f_n(t) = 0 \quad \text{for all } t \text{ in } I.$$

We can ~~talk a~~ characterize fundamental set of solutions of the homogeneous equation ~~by~~ in terms of linear independence.

Suppose that the functions $y_1, \cdots, y_n$ satisfies

$$k_1 y_1(t) + \cdots + k_n y_n(t) = 0 \quad \text{for all } t \text{ in } I.$$

We ~~then~~ can differentiate

$$k_1 y_1'(t) + \cdots + k_n y_n'(t) = 0.$$

$$\vdots$$

$$k_1 y_1^{(n-1)}(t) + \cdots + k_n y_n^{(n-1)}(t) = 0.$$

The determinant of the coefficient matrix is

$$W(y_1, \cdots, y_n)(t).$$

Theorem: If $y_1(t), \cdots, y_n(t)$ is a fundamental set of solutions of

$$y^{(n)} + p_1(t)\, y^{(n-1)} + \cdots + p_{n-1}(t)\, y' + p_n(t) \cdot y = 0.$$

on an interval I, then $y_1(t), \cdots, y_n(t)$ are linearly independent on I.

Conversely, if $y_1(t), \cdots, y_n(t)$ are linearly independent solutions of the equation on I, then they form a fundamental set of solutions.

proof: Suppose $y_1, \cdots, y_n$ form a fundamental set of solutions, then the Wronskian $W(y_1, \cdots, y_n)(t_0) \neq 0$ for at least one t_0 in I.

Then if $k_1 y_1 + \cdots + k_n y_n = 0 \implies k_1 = \cdots = k_n = 0$

Thus $y_1, \cdots, y_n$ are linearly independent.

Suppose ~~$y_1, \cdots, y_n$ for~~ $y_1, \cdots, y_n$ are linearly independent, we need to show that they ~~are~~ form a fundamental set of solutions, i.e., at least at 1-point t_0, the Wronskian: $W(y_1, \cdots, y_n)(t_0) \neq 0$.

Assume this is not true. For any $t_0 \in I$, $W(y_1, \cdots, y_n)(t_0) = 0$.

Then the equation

$$k_1 y_1(t_0) + \cdots + k_n y_n(t_0) = 0.$$

$$\vdots$$

$$k_1 y_1^{(n-1)}(t_0) + \cdots + k_n y_n^{(n-1)}(t_0) = 0$$

has a nonzero solution.

$k_1^*, \ldots, k_n^*$,

then $k_1^* y_1(t) + \cdots + k_n^* y_n(t) \neq 0$ since they are linearly independent.

This is a solution of homogeneous equation and the initial condition $y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0$.

This contradicts the uniqueness; since $y \equiv 0$ is also a soln.

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Section 4.2: Homogeneous equations w/ constant coefficients:

Consider the nth order linear homogeneous ODE:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0.$$

From the knowledge of 2nd order linear ODE w/ constant coefficients, it's natural to expect that $y = e^{rt}$ is a soln.

$$a_0 (e^{rt})^{(n)} + a_1 (e^{rt})^{(n-1)} + \cdots + a_{n-1} (e^{rt})' + a_n e^{rt} = 0.$$

$$\parallel$$

$$(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n) \cdot e^{rt} = 0.$$

$$\Rightarrow a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

This equation (above r) is called the characteristic equation of the ODE.

A polynomial of degree n has n zeroes, say $r_1, r_2, \cdots, r_n$!
They could be real or complex.

The easiest possibility is all the roots of the characteristic equation are ~~equal~~ real and no two are equal, then we have n distinct ~~roots~~ solutions of the homogeneous ODE: $e^{r_1 t}, \cdots, e^{r_n t}$ and the general solution is

$$y = c_1 e^{r_1 t} + \cdots + c_n e^{r_n t}.$$

If ~~there are~~ the characteristic equation has distinct roots, some are complex ~~roots~~, they must occur in conjugate pairs $\lambda \pm i\mu$, since the coefficients of the equation are real numbers.

~~Still suppose~~ ~~Suppose~~ ~~Then $e^{\lambda + i\mu t}$ and $e^{\lambda - i\mu t}$ can be replace~~

Then there are solutions of the ODE:

$$e^{\lambda t} \cos \mu t, \quad e^{\lambda t} \sin \mu t,$$

Example: Find the general solution of

$$y^{(4)} - y = 0.$$

The characteristic equation is

$$r^4 - 1 = 0.$$

$$\cancel{(r^2-1)}\ (r+1)(r-1)(r^2+1) = 0.$$

$$r_1 = -1,\ r_2 = 1,\ r_3 = i,\ r_4 = -i.$$

So there are solutions

$$y_1(t) = e^{-t},\ y_2(t) = e^{t},\ y_3(t) = \cos t,\ y_4(t) = \sin t.$$

The general solution is given by

$$y = c_1 e^{-t} + c_2 e^{t} + c_3 \cdot \cos t + c_4 \sin t.$$

Repeated roots: For an equation of order n ~~$Z(r)=0$~~, if ~~the~~ a root $r = r_1$ of

$Z(r) = 0$ has multiplicity s ($s \le n$), then

$e^{r_1 t}$, $\cancel{t \cdot e}$ $t \cdot e^{r_1 t}$, $\ldots$, $t^{s-1} e^{r_1 t}$ are solutions of

the homogeneous ~~equation~~ ODE.

There could also be the possibility that a complex root ~~is~~ $\lambda+i\mu$ is repeated s times, then the complex conjugate is ~~also~~ $\lambda-i\mu$ is also repeated s times. Then the following functions are solutions of the homogeneous ODE, and they are linearly independent:

$$e^{\lambda t}\cos\mu t,\ e^{\lambda t}\sin\mu t,\ t\cdot e^{\lambda t}\cos\mu t,\ t\cdot e^{\lambda t}\sin\mu t,$$

$$\cdots,\ t^{s-1}e^{\lambda t}\cos\mu t,\ t^{s-1}e^{\lambda t}\sin\mu t.$$

Example: Find the general solutions of

$$y^{(4)}+2y''+y=0.$$

The characteristic equation is

$$r^4+2r^2+1=0.$$
$$\parallel$$
$$(r^2+1)^2=0,$$

so the roots are i, $-i$ (both repeated twice).

And the general solution is

$$y=C_1\cos t+C_2\sin t+C_3t\cos t+C_4t\sin t.$$

4.3. The method of Undetermined coefficients:

Similar to the 2nd order ODE's, to find solutions of a nonhomogeneous nth order linear equation w/ constant coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t). \quad \cancel{\text{const}},$$

we need to do two things:

(1). Find the general solution of the corresponding homogeneous equation

(2). Find a particular solution.

For (2), we ~~also~~ can also use the method of undetermined coefficients:

Example: Find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^t.$$

The homogeneous equation has characteristic equation

$$r^3 - 3r^2 + 3r - 1 = 0.$$

$$\Rightarrow (r-1)^3 = 0.$$

So the general solution of the homogeneous equation is

$$c_1 e^t + c_2 t e^t + c_3 t^2 e^t.$$

Recall that to find a particular solution $Y(t)$, we can start by assuming $Y(t) = Ae^t$. But this does not work since e^t is a solution of the homogeneous equation.

We can actually try $Y(t) = A \cdot t^3 \cdot e^t$.

then $\cancel{Y'''(t) - 3Y''(t)}$ $Y'(t) = 3At^2 \cdot e^t + A \cdot t^3 \cdot e^t$,

$$Y''(t) = 6Ate^t + 3At^2e^t + 3At^2e^t + A \cdot t^3 \cdot e^t$$

$$= 6A \cdot te^t + 6At^2e^t + A \cdot t^3e^t.$$

$$Y'''(t) = 6Ae^t + 6Ate^t + 12Ate^t + 6At^2e^t + 3At^2e^t + A \cdot t^3e^t$$

$$= 6Ae^t + 18Ate^t + 9At^2e^t + A \cdot t^3e^t.$$

$Y''' - 3Y'' + 3Y' - Y$

$$= 6Ae^t + \cancel{18Ate^t} + \cancel{9At^2e^t} + \cancel{At^3e^t} - \cancel{18Ate^t} - \cancel{18At^2e^t} - \cancel{3At^3e^t}$$

$$+ \cancel{9At^2e^t} + \cancel{3At^3e^t} - \cancel{At^3e^t} = 6Ae^t = 4e^t$$

$$\Rightarrow A = \frac{2}{3}, \quad Y(t) = \frac{2}{3} t^3 e^t.$$

Example: Find a particular solution of

$$y''' - 4y' = t + 3\cos t + e^{-2t}.$$

The characteristic equation is

$$r^3 - 4r = 0.$$

$$r(r+2)(r-2) = 0.$$

$$r_1 = 0, \; r_2 = 2, \; r_3 = -2.$$

$$y_c(t) = C_1 + C_2 e^{2t} + C_3 e^{-2t}.$$

To find a particular solution, we ~~write~~ consider the following equations:

$$y''' - 4y' = t, \quad y''' - 4y' = 3\cos t, \quad y''' - 4y' = e^{-2t}.$$

For the first one, a particular solution $Y_1(t)$ could be $A_0 t + A_1$, But the constant function is a solution of the homogeneous equation, so we multiply it by t.

$$Y_1(t) = t \cdot (A_0 t + A_1)$$

$$= A_0 t^2 + A_1 t.$$

$$Y_1'''(t) = 0, \quad Y_1''(t) = 2A_0 t + A_1$$

$\Rightarrow$ [illegible]

For the second equation we choose

$$Y_2(t) = B\cos t + C\sin t.$$

$$Y_2'(t) = -B\sin t + C\cos t$$

$$Y_2''(t) = -B\cos t - C\sin t$$

$$Y_2'''(t) = B\sin t - C\cos t.$$

$$Y_2'''(t) - 4Y_2'(t) = (B + 4B)\sin t.$$

$$+ (-C - 4C).\cos t$$

$$\Longrightarrow B = 0, \; C = -\frac{3}{5}.$$

For the 3rd equation, we choose

$$Y_3(t) = E \cdot t \cdot e^{-2t}. \quad (\text{sin } e^{-2t} \text{ does NOT work}).$$

$\cancel{Y_3'(t) =}$

You can find $E = \frac{1}{8}$

So a particular solution is

$$Y(t) = -\frac{1}{8}t^2 - \frac{3}{5}\sin t + \frac{1}{8}te^{-2t}.$$

The method of Variation of Parameters:

~~Suppose~~ Say the nonhomogeneous nth order linear ODE is

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y'(t) + p_n(t) = g(t).$$

Suppose we already know ~~the~~ a fundamental set of solutions of the corresponding homogeneous equation

$$y_c(t) = c_1 y_1(t) + \cdots + c_n y_n(t).$$

The method of variation of parameters is to let. c_i become functions instead of constants.

$$Y(t) = u_1(t)\cdot y_1(t) + \cdots + u_n(t)\, y_n(t).$$

$$Y'(t) = u_1'(t)\, y_1(t) + u_2'(t) y_2(t) + \cdots + u_n'(t)\, y_n(t)$$

$$u_1(t)\, y_1'(t) + u_2(t) y_2'(t) + \cdots + u_n(t)\, y_n'(t)$$

We need to impose the condition that.

$$u_1'(t) y_1(t) + u_2'(t) y_1(t) + \cdots + u_n'(t)\, y_n(t) = 0.$$

And we get. $Y'(t) = u_1(t)\, y_1'(t) + \cdots + u_n(t)\, y_n'(t).$

$$Y''(t) = u_1'(t)\, y_1'(t) + \cdots + u_n'(t)\, y_n'(t)$$

$$+ u_1(t)\, y_1''(t) + \cdots + u_n(t)\, y_n''(t).$$

Again we impose the condition.

$$u_1'(t)\, y_1'(t) + \cdots + u_n'(t)\, y_n'(t) = 0.$$

This can be repeated.

$$u_1'(t)\, y_1^{(n-1)}(t) + \cdots + u_n'(t)\, y_n'(t) = 0.$$

$$Y^{(n)}(t) = \underbrace{u_1'(t)\, y_1^{(n-1)}(t) + \cdots + u_n'(t)\, y_n^{(n-1)}(t)}_{} = 0.$$

$$+ u_1(t)\, y_1^{(n)}(t) + \cdots + u_n(t)\, y_n^{(n)}(t).$$

And $$Y^{(n)}(t) = u_1'(t)\, y_1^{(n-1)}(t) + \cdots + u_n'(t)\, y_n^{(n-1)}(t)$$

$$+ u_1(t)\, y^{(n)}(t) + \cdots + u_n(t)\, y_n^{(n)}(t).$$

And we find

$$Y^{(n)}(t) + p_1(t)\, Y^{(n-1)}(t) + \cdots + p_{n-1}(t)\, Y'(t) + p_n(t)\, Y(t)$$

$$= u_1'(t)\, y_1^{(n-1)}(t) + \cdots + u_n'(t)\, y_n^{(n-1)}(t) = g(x).$$

To generalize, we have

$$y_1 u_1' + \cdots + y_n u_n' = 0.$$
$$\vdots$$
$$y_1^{(n-2)} u_1' + \cdots + y_n^{(n-2)} u_n' = 0.$$
$$y_1^{(n-1)} u_1' + \cdots + y_n^{(n-1)} u_n' = g.$$

$$\Longrightarrow u_m'(t) = \frac{g(t) W_m(t)}{W(t)} \qquad m = 1, 2, \cdots, n.$$

(This follows from Cramer's rule).

Example: Given that $y_1(t) = e^t$, $y_2(t) = te^t$, $y_3(t) = e^{-t}$ are solutions of the homogeneous equation corresponding to

$$y''' - y'' - y' + y = g(t).$$

determine a particular solution in terms of an integral.

$$W(t) = \begin{vmatrix} e^t & te^t & e^{-t} \\ e^t & (t+1)e^t & -e^{-t} \\ e^t & (t+2)e^t & e^{-t} \end{vmatrix} = e^t \begin{vmatrix} 1 & t & 1 \\ 1 & t+1 & -1 \\ 1 & t+2 & 1 \end{vmatrix}.$$

$$= e^t \begin{vmatrix} 1 & t & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 2 \end{vmatrix} = e^t \cdot 4.$$

$$W_1(t) = \begin{vmatrix} 0 & te^t & e^{-t} \\ 0 & (t+1)e^t & -e^{-t} \\ 1 & (t+2)e^t & e^{-t} \end{vmatrix} = -2t-1.$$

$$W_2(t) = \begin{vmatrix} e^t & 0 & e^{-t} \\ e^t & 0 & -e^{-t} \\ e^t & 1 & e^{-t} \end{vmatrix} = 2$$

$$W_3(t) = \begin{vmatrix} e^t & te^t & 0 \\ e^t & (t+1)e^t & 0 \\ e^t & (t+2)e^t & 1 \end{vmatrix} = e^{2t}.$$

$$Y(t) = \overset{y_1(t)}{\overset{\parallel}{e^t}} \cdot \int_{t_0}^{t} \frac{g(s)(-1-2s)}{4e^s}\, ds$$

$$+ te^t \int_{t_0}^{t} \frac{g(s) \cdot 2}{4 \cdot e^s}\, ds.$$

$$+ \; e^{-t} \int_{t_0}^{t} \frac{g(s) \cdot e^{2s}}{4 \cdot e^s}\, ds.$$